Odd Dimensional Symplectic Manifolds

by

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Bachelor of Sciences, Peking University, China 2005

Submitted to the Department of Mathematics
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

June 2010

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April 16, 2010

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Abstract

In this thesis, we introduce the odd dimensional symplectic manifolds. In the first half we study the Hodge theory on the basic symplectic manifolds. We can define two cohomology theories on them, the standard basic de Rham cohomology gheory and a basic version of the Koszul-Brylinski-Mathieu 'harmonic' symplectic cohomology theory. Among our main results are a collection of examples for which these cohomology theories don't coincide, and, in fact, for which the usual basic cohomology theory is infinite dimensional and the symplectic cohomology theory is finite dimensional. On the other hand, we prove an odd version of the Mathieu theorem and the $d\delta$-lemma: the two theories coincide if and only if a basic version of strong Lefschetz property holds. In the second half, we discuss the group actions on odd dimensional symplectic manifolds. In particular, we study the Hamiltonian group actions. Finally we use the Local-Global-Principle to prove a convexity theorem for the Hamiltonian torus actions on odd dimensional symplectic manifolds.

Thesis Supervisor: Victor W. Guillemin
Title: Professor of Mathematics

Acknowledgments

First of all I would like to thank my thesis advisor, Victor Guillemin. Without his inspiration and guidance, I would not have survived the last five years at MIT. Without his ideas and advice, this thesis would never have been written.

I wish to thank my parents, my mom Guofang and my dad Liangsheng, for raising me, educating me and providing me with the best education environment. Without all these, I would never have accomplished what I have today.

Last but not the least, I am deeply indebted to my dear wife Yuan, for her consistent support, and for giving birth to our lovely newborn baby boy Aidan.

Contents

Introduction

The topic of this thesis is "odd dimensional" symplectic geometry. In more detail: an odd dimensional symplectic manifold is a $(2n+1)$-dimensional manifold, M, with a closed two-form ω of maximal rank, and a volume form Ω. Thus it possesses a canonical Reeb vector field, R, with the defining properties

$$\iota(R)\omega = 0,\ \iota(R)\Omega = \frac{\omega^n}{n!}.$$

We will also for some applications assume that there exists a 'connection', i.e., a one-form λ with the properties

$$\iota(R)\lambda = 1,\ \mathcal{L}(R)\lambda = 0,$$

in which case our volume form is $\lambda \wedge \frac{\omega^n}{n!}$.

If the flow of R is fibrating, the quotient, X, of M by this flow is itself a symplectic manifold, and the "odd dimensional" symplectic geometry of M is essentially the symplectic geometry of X. The question we address in this thesis is: What can one say when the Reeb flow is not fibrating? In this case one can define two cohomology theories on X, the standard basic De Rham cohomology theory of M and a basic version of the Koszul-Brylinski-Mathieu "harmonic" symplectic cohomology theory. Among our main results are a collection of examples for which these cohomology theories *don't* coincide, and, in fact, for which the usual basic cohomology theory is infinite dimensional and the symplectic theory is finite dimensional. On the other hand, we prove an odd version of the Mathieu theorem: the two theories coincide if and only if a basic version of hard Lefschetz property holds and also prove an odd version of the $d\delta$ lemma which shows that a spectral sequence for the basic symplectic harmonic cohomology degenerates at its E_2 stage if the basic version of hard Lefschetz holds. In the second half of the thesis we take up the question of group actions on odd symplectic manifolds and prove a number of results that extend to the odd case classical theorems about Hamiltonian actions of Lie groups.

We will now give a more detailed section by section description of the thesis. In chapter 1, we summarize basic facts about odd dimensional symplectic manifolds. In particular we prove a Darboux Theorem which asserts that locally in the neighborhood of any point $p \in M$ there exist coordinates $x_1, y_1, \cdots x_n, y_n, z$ such that

$$\omega = dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n,$$

$$\Omega = dz \wedge dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n,$$

$$R = \frac{\partial}{\partial z}.$$

We also describe a number of standard examples of odd dimensional symplectic manifolds. Among them are contact manifolds, mapping tori and Poisson manifolds with codimension one symplectic leaves and circle bundles over symplectic manifolds.

In chapter 2 we discuss basic Hodge theoretic properties of the symplectic quotient of M by the Reeb flow, an object which we will denote by X even though in general it is not a manifold or even Hausdorff topological space. Its De Rham complex, however, is well defined as the basic De Rham complex of M. Among other things, we will define the Hodge star operator on this basic complex, prove some basic Hodge theoretic identities, show how to define its symplectic Hodge theory, and discuss the Weil representation of $sl(2)$ on $\Omega(X) = \Omega_{\mathsf{bas}}(M)$.

In chapter 3 we describe the strong Lefschetz property and the basic version of the notion of primitivity, and finally state and prove Mathieu's theorem and the $d\delta$ lemma. We will next exhibit the counterexamples mentioned above in section 3.5: odd dimensional symplectic manifolds M for which the usual De Rham cohomology of X is infinite dimensional but the symplectic cohomology is finite dimensional.

As mentioned above, the second half of the thesis will be focussed on results having to do with Hamiltonian actions of groups on odd dimensional symplectic manifolds. In chapter 4 we will (using the Local-Global-Principle, which we will describe in the appendix) prove that the Atiyah-Guillemin-Sternberg convexity theorem has an odd dimensional analogue.

Chapter 1

Odd Dimensional Symplectic Manifolds

In this chapter, we define symplectic structures on odd dimensional manifolds. The first section discusses elementary properties of odd dimensional sympletic vector spaces. The second section defines what an odd dimensional symplectic manifold is, and proves a Darboux theorem, and the third section discusses some basic examples of odd dimensional symplectic manifolds.

1.1 Odd Dimensional Symplectic Vector Space

Let V be a $(2n+1)$-dimensional vector space, $\omega \in \Lambda^2(V^*)$ a 2-covector (antisymmetric bilinear form), and $\Omega \in \Lambda^{2n+1}(V^*)$ a volume form.

Definition 1.1.1. *We say that the triple (V, ω, Ω) is an odd dimensional symplectic vector space if the kernel of ω is 1-dimensional, or equivalently $\omega^n \neq 0$.*

Note that there is a canonical vector r defined by $\iota(r)\omega = 0$ and $\iota(r)\Omega = \frac{\omega^n}{n!}$. Pick any complimentary subspace W to the 1-dimensional subspace generated by r, ω is a symplectic form when restricted to W. Therefore, we have a Darboux basis $e_1, f_1, \cdots, e_n, f_n$ for W. Note that $e_1, f_1, \cdots, e_n, f_n, r$ is a basis for V, so we have the dual basis $e^1, f^1, \cdots, e^n, f^n, r^*$ for V^*. It is quite clear that

$$\omega = \sum_{i=1}^{n} e^i \wedge f^i$$

since the right hand side of the equation has the 1-dimensional kernel generated by R as well, and is equal to ω when restricted to W. Furthermore, the property $\iota(r)\Omega = \frac{\omega^n}{n!}$ forces that

$$\Omega = e^1 \wedge f^1 \wedge \cdots e^n \wedge f^n \wedge r^*.$$

We conclude this in the following theorem.

Theorem 1.1.2 (Darboux Basis). *Suppose (V, ω, Ω) is a $(2n+1)$-dimensional symplectic vector space, then there exists a basis $e_1, f_1, \cdots, e_n, f_n, r$ of V and dual basis $e^1, f^1, \cdots, e^n, f^n, r^*$ of V^*, such that*

$$\omega = \sum_{i=1}^{n} e^i \wedge f^i,$$

$$\Omega = e^1 \wedge f^1 \wedge \cdots \wedge e^n \wedge f^n \wedge r^*.$$

Remark 1.1.3. *By the discussion above, we can see that we have a lot of choices of W. However, the subspace generated by $e^1, f^1, \cdots, e^n, f^n$ is independent of the choice of the complimentary subspace W, since it is actually the annihilator of r.*

1.2 Odd Dimensional Symmplectic Manifold

Suppose that M is a manifold of dimension $2n+1$, ω is a closed 2-form of maximal rank, i.e., $\omega^n \neq 0$ everywhere, and Ω is a volume form. We will call $\sigma := \frac{\omega^n}{n!}$ basic volume form. Since ω is of maximal rank, $\ker \omega$ is a 1-dimensional foliation on M. Moreover, there is a canonical vector field R defined by

$$\iota(R)\omega = 0$$

$$\iota(R)\Omega = \frac{\omega^n}{n!}$$

We say R is the Reeb vector field on M. With this foliation, we can define horizontal forms, invariant forms and basic forms as follows.

$$\Omega_{\text{hor}}(M) = \{\alpha \in \Omega(M) |\ \iota(R)\alpha = 0\}$$

$$\Omega_{\text{inv}}(M) = \{\alpha \in \Omega(M) |\ \mathcal{L}(R)\alpha = 0\}$$

$$\Omega_{\text{bas}}(M) = \{\alpha \in \Omega(M) |\ \iota(R)\alpha = 0, \mathcal{L}(R)\alpha = 0\}$$

Definition 1.2.1 (Odd Dimensional Symplectic Manifold). *Suppose that M is a manifold of dimension $2n+1$ with a volume form Ω and a closed 2-form ω of maximal rank. Then the triple (M, ω, Ω) is called an odd-dimensional symplectic manifold. Equivalently, an odd dimensional symplectic manifold is a triple (M, ω, Ω) such that: (1) ω is a closed form; (2) for any $p \in M$ the triple $(T_pM, \omega_p, \Omega_p)$ is a symplectic vector space.*

Remark 1.2.2. *In an earlier version of the definition, we required there to exist an R-invariant 1-form $\lambda \in \Omega^1_{inv}(M)$, such that $\iota(R)\lambda = 1$, where we called λ the connection 1-form on M. Such a connection 1-form exists in almost all the examples we will discuss in this thesis. However, almost none of the results in chapters 1, 2 and 3 rely on the existence of this connection 1-form. We note that if λ exists, $\lambda \wedge \frac{\omega^n}{n!}$.*

Indeed, since $\iota(R)\omega = 0$, $\iota(R)(\lambda \wedge \sigma) = (\iota(R)\lambda) \wedge \sigma$. Therefore,

$$\begin{aligned} \iota(R)\lambda = 1 &\Leftrightarrow \iota(R)(\lambda \wedge \sigma) = \sigma \\ &\Leftrightarrow \iota(R)(\lambda \wedge \sigma) = \iota(R)\Omega \\ &\Leftrightarrow \Omega = \lambda \wedge \sigma. \end{aligned}$$

We will now discuss the local structure of an odd dimensional symplectic manifold. Since R is a nowhere vanishing vector field, locally we can choose coordinates $(z_1, z_2, \cdots, z_{2n}, z)$ such that $\frac{\partial}{\partial z} = R$. In other words, locally the odd dimensional symplectic manifold is just $W \times (-\varepsilon, \varepsilon)$, where W has local coordinates $z_1, \cdots, z_{2n}$ and $(-\varepsilon, \varepsilon)$ has local coordinate z. Note that ω is a basic 2-form (i.e. $\iota(R)\omega = 0$ and $\mathcal{L}(R)\omega = 0$), so ω is actually a 2-form on W.

In addition, it is obvious that ω is non-degenerate on W, so by the Darboux theorem, we can choose another set of coordinates $x_1, y_1, \cdots, x_n, y_n$ for W, such that $\omega = dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n$. Since locally M is the Cartesian product $W \times (-\varepsilon, \varepsilon)$, $x_1, y_1, \cdots, x_n, y_n, z$ also serve as local coordinates for M.

Moreover, since $\iota(R)\Omega = \frac{\omega^n}{n!}$, we have $\Omega = dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n \wedge dz$. We thus get the following Darboux theorem for odd dimensional symplectic manifolds.

Theorem 1.2.3 (Darboux Coordinates). *Suppose that (M, ω, Ω) is an $(2n+1)$-dimensional symplectic manifold, then for any $p \in M$ there exists a neighborhood U of p and local coordinates $x_1, y_1, \cdots, x_n, y_n, z$ such that*

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i,$$

$$\Omega = dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n \wedge dz,$$

$$R = \frac{\partial}{\partial z}.$$

1.3 Examples

In this section, we describe three examples of odd dimensional symplectic manifolds that are often encountered in practice.

Example 1.3.1 (S^1-bundles over symplectic manifolds). *Suppose that (X, ω_0) is a symplectic manifold of dimension $2n$, and $\pi : M \to X$ is an S^1-bundle and θ is a connection 1-form on the S^1-bundle. Then we can equip M with an odd dimensional* *symplectic structure by letting*

$$\omega = \pi^*\omega_0$$

be the symplectic 2*-form and*

$$\Omega = \theta \wedge \frac{\omega^n}{n!}$$

be the volume form. Moreover, the connection form θ *of the circle bundle serves as a connection form for* M*. One can check that the Reeb vector field* R *is generated by the* S^1*-action on* M*. Note that the volume form* Ω *above does not depend on the choice of connection form* θ*, since the difference of any two connection* 1*-form is horizontal.*

Example 1.3.2 (Contact Manifolds). *Any contact manifold with a contact* 1*-form* α *can be viewed as an odd-dimensional symplectic manifold in the obvious way:*

$$\omega := d\alpha$$

is the symplectic 2*-form,* α *is the connection form, and hence the volume form is*

$$\Omega = \alpha \wedge \frac{(d\alpha)^n}{n!}$$

Example 1.3.3 (Mapping Tori of Symplectic Transformations). *Suppose that* (X, ω_0) *is a symplectic manifold of dimension* $2n$*, and*

$$\varphi : X \to X$$

is a symplectic transformation (i.e., φ *is an diffeomorphism such that* $\varphi^*\omega_0 = \omega_0$*). We define the mapping torus of* φ *as follows. The map* φ *induces a diffeomorphism*

$$f_\varphi : X \times \mathbb{R} \to X \times \mathbb{R}$$

defined by $f_\varphi(x,t) = (\varphi(x), t+1)$*. Let* Γ *be the group of diffeomorphisms generated* *by f_φ, then Γ acts freely on $X \times \mathbb{R}$, and the mapping torus of φ is defined by*

$$M_\varphi := (X \times \mathbb{R})/\Gamma.$$

Equivalently, $M_\varphi = (X \times [0,1])/\sim$, where the equivalence relation is defined by $(x,0) \sim (\varphi(x),1)$ for all $x \in X$. Note that M_φ is a $(2n+1)$-dimensional manifold.

We equip M_φ with a symplectic structure as follows. First, we pull ω_0 back by the projection map ($X \times \mathbb{R} \to X$) to get a closed 2-form on $X \times \mathbb{R}$. Since φ is a symplectic transformation, i.e. ω_0 is φ-invariant, the pull-back 2-form is Γ-invariant. Hence, it induces a closed 2-form ω on M_φ. Similarly, The canonical 1-form dt on $\mathbb{R}$ can be pulled back to $X \times \mathbb{R}$, and it is also φ-invariant, so dt induces to be a closed 1-form on M_φ as well.

Note that $X \times \mathbb{R}$ is a trivial odd dimensional symplectic manifold, and the mapping torus M_φ is just a discrete quotient of $X \times \mathbb{R}$ with symplectic form ω_0 and connection form dt. Therefore, the forms induced by them on M_φ can serve as the symplectic 2-form and the connection 1-form on M_φ. Note that the Reeb vector field on $X \times \mathbb{R}$ is just $\frac{\partial}{\partial t}$. One can check that the vector field $\frac{\partial}{\partial t}$ on $X \times \mathbb{R}$ is preserved by f_φ as well, and hence reduces to be the Reeb vector field on M_φ.

Remark 1.3.4. *We conclude by observing that the contact manifolds and the mapping tori of symplectic transformations are two opposite extremes of odd-dimensional symplectic manifolds. Contact manifolds have connection 1-forms, whose exterior derivatives have maximal rank; while the mapping tori of symplectic transformations have closed connection 1-forms, i.e. their exterior derivatives are zero.*

Chapter 2

Odd Dimensional Symplectic Hodge Theory

2.1 Hodge Theory on Odd Dimensional Symplectic Vector Space

2.1.1 Star Operator on Odd Dimensional Symplectic Vector Spaces

Suppose (V, ω, Ω) is a $(2n+1)$-dimensional symplectic vector space. Recall that there is a canonical vector $r \in V$, and a Darboux basis $e_1, f_1, \cdots, e_n, f_n, r$ for V such that

$$\omega = \sum_{i=1}^{n} e^i \wedge f^i.$$

Let $e^1, f^1, \cdots, e^n, f^n, r^*$ be the dual basis for V^*. Then the annihilator of r is just the subspace generated by $e^1, f^1, \cdots, e^n, f^n$, i.e.

$$\mathrm{ann}\{r\} = \langle e^1, f^1, \cdots, e^n, f^n \rangle.$$

Since the kernel of ω is the 1-dimensional subspace $\langle r \rangle$, we can regard ω as a bilinear form on $V/\langle r \rangle$. On the other hand, the annihilator $\mathrm{ann}\{r\}$ is isomorphic to the dual space $(V/\langle r\rangle)^*$ in an obvious way. Moreover, there is a natural isomorphism between $V/\langle r\rangle$ and its dual space $(V/\langle r\rangle)^*$. To sum up,

$$V/\langle r\rangle \cong (V/\langle r\rangle)^* = \mathrm{ann}\{r\},$$

then there is a natural isomorphism between $V/\langle r\rangle$ and $\mathrm{ann}\{r\}$. Therefore, the antisymmetric nondegenerate bilinear form ω on $V/\langle r\rangle$ induces an antisymmetric nondegenerate bilinear form B on $\mathrm{ann}\{r\}$. For convenience, we denote the annihilator space $\mathrm{ann}\{r\}$ by N.

Lemma 2.1.1. *Suppose Σ is an antisymmetric nondegenerate bilinear form on a $2n$-dimensional vector space W, then it can be extended to be a nondegenerate bilinear form on $\Lambda^p(W)$ for all $p = 0, 1, \cdots, 2n$.*

Proof. We can first extend Σ over these decomposable p-forms:

$$\Sigma(\mu_1 \wedge \mu_2 \wedge \cdots \wedge \mu_p, \nu_1 \wedge \nu_2 \wedge \cdots \wedge \nu_p) := \det[\Sigma(\mu_i, \nu_j)]_{i,j},$$

where $[B(\mu_i, \nu_j)]_{i,j}$ is the square matrix whose (i, j) entry is $B(\mu_i, \nu_j)$. Next, we only need extend B bilinearly over all of the $\Lambda^p(W)$. This is well-defined because of the linearity of determinant of matrices.

We next prove Σ is nondegenerate on $\Lambda^p(W)$. Note that Σ is actually a symplectic 2-form on W, we can find a Darboux basis $\mu_1, \nu_1, \cdots, \mu_n, \nu_n$ such that $\Sigma = \sum_{i=1}^n \mu^i \wedge \nu^i$ where $\mu^1, \nu^1, \cdots, \mu^n, \nu^n$ is the dual basis. Thus, $B(\mu_i, \mu_j) = B(\nu_i, \nu_j) = 0$ and $\Sigma(\mu_i, \nu_j) = \delta_{ij}$. Suppose $\alpha \in \Lambda^p(W)$ is in the kernel of Σ. We can certainly write α as a linear combination of decomposable p-forms:

$$\alpha = \sum_{0\le k\le p,\ 1\le i_1\le\cdots\le i_k\le n,\ 1\le j_{k+1}\le\cdots\le j_p\le n} a_{i_1,\cdots,i_k,j_{k+1},\cdots,j_p}\mu_{i_1} \wedge \cdots \wedge \mu_{i_k} \wedge \nu_{j_{k+1}} \wedge \cdots \wedge \nu_{j_p}.$$

Then $0 = \Sigma(\alpha, \nu_{i_1} \wedge \cdots \wedge \nu_{i_k}) \wedge (-\mu_{j_{k+1}}) \wedge \cdots \wedge (-\mu_{j_p}) = a_{i_1,\cdots,i_k,j_{k+1},\cdots,j_p}$. Therefore, $\alpha = 0$. This completes the proof of nondegeneracy of Σ on $\Lambda^p(W)$. $\square$

Therefore, B can be extended to be a nondegenerate form on $\Lambda^p(N)$. Now we are ready to define the Hodge star operator on $\Lambda^*(N)$.

Definition 2.1.2. *Suppose that (V, ω, Ω) is an $2n+1$-dimensional symplectic vector space, and r is the vector mentioned before. For $p = 0, 1, \cdots, 2n$, the star operator on the annihilator space N of r*

$$\star : \Lambda^p(N) \longrightarrow \Lambda^{2n-p}(N)$$

is a linear map with the following defining property:

$$u \wedge (\star v) = B(u, v)\frac{\omega^n}{n!}$$

for all $u, v \in \Lambda^p(N)$.

Recall that A has a basis $e^1, f^1, \cdots, e^n, f^n$, and $B(e^i, e^j) = \omega(-f_i, -f_j) = 0$, $B(f^i, f^j) = \omega(e_i, e_j) = 0$, $B(e^i, f^j) = \omega(-f_i, e_j) = \delta_{ij}$.

Let N_k be the subspace generated by e^k, f^k, $B_k = B|_{A_k}$ the bilinear form B restricted to N_k, and $\Omega_k = e^k \wedge f^k$. Then we can also define a star operator $\star_k$ on $(N_k, \omega_k, \Omega_k)$. By definition, $1 \wedge (\star_k 1) = B_k(1, 1)e^k \wedge f^k = e^k \wedge f^k$, so $\star_k 1 = e^k \wedge f^k$; $e^k \wedge (\star_k f_k) = B_k(e_k, f_k)e_k \wedge f_k = e_k \wedge f_k$ and $f^k \wedge (\star_k f^k) = B_k(f_k, f_k)e^k \wedge f^k = 0$, so $\star_k f^k = f^k$; similarly, we have $\star_k e^k = e^k$; $e_k \wedge f_k \wedge \star_k(e^k \wedge f^k) = B_k(e^k \wedge f^k, e^k \wedge f^k)e^k \wedge f^k = e^k \wedge f^k$, so $\star_k(e^k \wedge f^k) = 1$. In particular, $\star_k^2 = \mathrm{id}$.

Lemma 2.1.3. *Let $\alpha_k \in \{1, e^k, f^k, e^k \wedge f^k\}$, $k = 1, 2, \cdots, n$. Then*

$$\star(\alpha_1 \wedge \cdots \wedge \alpha_n) = (\star_n \alpha_n) \wedge \cdots \wedge (\star_1 \alpha_1).$$

In particular, $\star^2 = id$.

Proof. Let $\beta_k \in \{1, e^k, f^k, e^k \wedge f^k\}$ such that $|\alpha_k| = |\beta_k|$, $k = 1, 2, \cdots, n$. Then

$$\begin{aligned}
\beta_1 \wedge \cdots \wedge \beta_n \wedge \star(\alpha_1 \wedge \cdots \wedge \alpha_n) &= B(\beta_1 \wedge \cdots \wedge \beta_n, \alpha_1 \wedge \cdots \wedge \alpha_n)\frac{\omega^n}{n!} \\
&= \Pi_{k=1}^n (B_k(\beta_k, \alpha_k)\Omega_k) \\
&= \Pi_{k=1}^n (\beta_k \wedge \star_k \alpha_k) \\
&= \beta_1 \wedge \cdots \wedge \beta_n \wedge \star_n \alpha_n \wedge \cdots \wedge \star_1 \alpha_1.
\end{aligned}$$

Note that every form is a linear combination of $\beta_1 \wedge \cdots \wedge \beta_n$-type forms, so the equalities above complete the proof of $\star(\alpha_1 \wedge \cdots \wedge \alpha_n) = (\star_n \alpha_n) \wedge \cdots \wedge (\star_1 \alpha_1)$. And for the same reason, $\star^2 = \mathrm{id}$ since it is true on all such type of forms as discussed above. □

2.1.2 Some Hodge Identities

Let (V, ω, Ω) be a $(2n+1)$-dimensional symplectic vector space and let

$$\star : \Lambda^p(N) \to \Lambda^{2n-p}(N)$$

and

$$E : \Lambda^p(N) \to \Lambda^{p+2}(N)$$

be the star operator and the operator multiplication by ω. E is well defined since $\omega \in \Lambda^2(N)$. Conjugating E with $\star$ one gets the transpose operator

$$F = \star E \star : \Lambda^p(N) \to \Lambda^{p-2}(N).$$

Lemma 2.1.4. *Let $e_1, f_1, \cdots, e_n, f_n, r$ be a Darboux basis for the $(2n+1)$-dimensional symplectic vector space (V, ω, Ω), then for any $u \in \Lambda^*(N)$,*

$$Fu = \sum_{k=1}^n \iota(f_k)\iota(e_k)u.$$

Proof. We only need to check the equation on $\alpha_1 \wedge \cdots \wedge \alpha_n$, where $\alpha_k \in \{1, e^k, f^k, e^k \wedge f^k\}$, $k = 1, 2, \cdots, n$. So

$$\begin{aligned}
F(\alpha_1 \wedge \cdots \wedge \alpha_n) &= \star E \star (\alpha_1 \wedge \cdots \wedge \alpha_n) \\
&= \star E(\star_n \alpha_n \wedge \cdots \wedge \star_1 \alpha_1) \\
&= \sum_{k=1}^{n} \star[\star_n \alpha_n \wedge \cdots \wedge (e^k \wedge f^k \wedge \star_k \alpha_k) \wedge \cdots \wedge \star_1 \alpha_1] \\
&= \sum_{k=1}^{n} \alpha_1 \wedge \cdots \wedge \star_k (e^k \wedge f^k \wedge \star_k \alpha_k) \wedge \cdots \wedge \alpha_n \\
&= \sum_{k=1}^{n} \alpha_1 \wedge \cdots \wedge (\iota(f_k)\iota(e_k)\alpha_k) \wedge \cdots \wedge \alpha_n \\
&= \sum_{k=1}^{n} \iota(f_k)\iota(e_k)(\alpha_1 \wedge \cdots \wedge \alpha_n).
\end{aligned}$$

□

Lemma 2.1.5. *$[E, F] = H$, where for $u \in \Lambda^p(N)$, $p = 0, 1, \cdots, 2n$,*

$$Au = (p - n)u.$$

Moreover, $[H, E] = 2E$ and $[H, F] = -2F$.

Proof. By Lemma 2.1.4, $F = \sum \iota(f_k)\iota(e_k)$. And $E = \sum e^k \wedge f^k$. Let $u \in \Lambda^p(N)$, then

$$\begin{aligned}
(EF - FE)u &= \sum_{i=1}^{n}\sum_{j=1}^{n}[(e^i \wedge f^i) \wedge \iota(f_j)\iota(e_j)u - \iota(f_j)\iota(e_j)(e^i \wedge f^i \wedge u)] \\
&= \sum_{i=1}^{k}[(e^k \wedge f^k) \wedge \iota(f_k)\iota(e_k)u - \iota(f_k)\iota(e_k)(e^k \wedge f^k \wedge u)].
\end{aligned}$$

Note in the last equality we use that all the terms with $i \neq j$ is zero. Consider $\beta_k := (e^k \wedge f^k) \wedge \iota(f_k)\iota(e_k)\alpha_k - \iota(f_k)\iota(e_k)(e^k \wedge f^k \wedge \alpha_k)$ where $\alpha_k \in \{1, e^k, f^k, e^k \wedge f^k\}$. It is easy to check that $\beta_k = -1$ if $\alpha_k = 1$; $\beta_k = 0$ if $\alpha_k = e^k, f^k$; $\beta_k = e^k \wedge f^k$ if $\alpha_k = e^k \wedge f^k$. To sum up, $\beta_k = (|\alpha_k| - 1)\alpha_k$ where $|\alpha_k|$ is the degree of α_k. Now let's get back to the summation above: we only need to check the formula for $\alpha_1 \wedge \cdots \wedge \alpha_n$

where $\sum_{k=1}^{n}|\alpha_k| = p$,

$$\begin{aligned}
(EF-FE)(\alpha_1\wedge\cdots\wedge\alpha_n) &= \sum_{i=1}^{k}[(e^k\wedge f^k)\wedge\iota(f_k)\iota(e_k)-\iota(f_k)\iota(e_k)(e^k\wedge f^k)](\alpha_1\wedge\cdots\wedge\alpha_n)\\
&= \sum_{k=1}^{n}(\alpha_1\wedge\cdots\wedge\beta_k\wedge\cdots\wedge\alpha_n)\\
&= \sum_{k=1}^{n}(|\alpha_k|-1)(\alpha_1\wedge\cdots\wedge\alpha_n)\\
&= (p-n)\alpha_1\wedge\cdots\wedge\alpha_n.
\end{aligned}$$

Therefore, $[E,F]=H$. On the other hand, $[H,E]=2E$ and $[H,F]=-2F$ are obvious. □

2.1.3 Representation of $sl(2)$

Let (V,ω,Ω) be a $(2n+1)$-dimensional symplectic vector space. In the Lemma 2.1.5, we prove that $[E,F]=H$, $[H,E]=2E$ and $[H,F]=-2F$, so there is a representation of $sl(2)$ on the annihilator space N. This will be an important tool when we prove Mathieu's theorem and the $d\delta$-Lemma in chapter 3. Therefore, we digress a bit in this subsection to go through some well-known facts about the representation of $sl(2)$.

Suppose that there is a representation of $sl(2)$ on a vector space W. We still use E, F and H to denote the basis of $sl(2)$. Let $v\in W$ be an eigenvector of H with eigenvalue l, then it is clear that Ev and Fv are both eigenvectors with eigenvalues $l+2$ and $l-2$ respectively.

Definition 2.1.6. *Suppose that there is a representation of $sl(2)$ on the vector space W, which is possibly infinite dimensional. We say that W is an $sl(2)$-module of $finite$ H-type if:*

1. *W can be decomposed as direct sum of eigenspaces of H;*

2. *H only has finitely many distinct eigenvalues.*

Theorem 2.1.7. *Suppose that (M,ω,Ω) is an odd dimensional symplectic manifold, then $\Omega_{bas}(M)$ is an $sl(2)$-module of finite H-type.*

From now on, we assume the representation of $sl(2)$ is of finite H-type. All these results can be applied to the odd dimensional symplectic spaces since they are automatically of finite H-type.

Lemma 2.1.8. *Let $v \in W$ be an eigenvector of H with eigenvalue λ, i.e. $Hv = \lambda v$. Then*

$$[E, F^k]v = k(\lambda - k + 1)F^{k-1}v$$

Proof. When $k = 1$, it is just $[E, F]v = \lambda v = Hv$. Suppose it is true for $k - 1$, then

$$\begin{aligned}
[E, F^k]v &= EF^k v - F^k Ev \\
&= EF^{k-1}Fv - F^k Ev \\
&= (F^{k-1}E + (k-1)((\lambda - 2) - (k-1) + 1)F^{k-2})Fv - F^k Ev \\
&= F^{k-1}(FE + H)v + (k-1)(\lambda - k)F^{k-1}v - F^k Ev \\
&= k(\lambda - k + 1)F^{k-1}v.
\end{aligned}$$

Note that the third equality uses the induction hypothesis. □

Lemma 2.1.9. *Let $v \in W$ and let W' be the $sl(2)$-submodule generated by v. Then W' is a finite dimensional space.*

Proof. By the definition of finite H-type, we can write

$$v = v_1 + v_2 + \cdots + v_m,$$

where $Hv_i = \lambda_i v_i$. We only need to prove that the submodule generated by v_i is finite dimensional. By Lemma 2.1.8, we know that any element in the submodule can be written as a linear combination of $E^j F^k v_i$. Moreover, since the submodule is of finite H-type, there are finitely many nontrivial $E^j F^k v_i$. Therefore, the submodule generated by v_i is finite dimensional. □

Corollary 2.1.10. *Every irreducible $sl(2)$-module of finite H-type is finite dimensional. Every cyclic $sl(2)$-module of finite H-type is a finite direct sum of irreducibles.*

Let's first investigate the structure of an irreducible $sl(2)$-module. Suppose that W is an irreducible $sl(2)$-module of finite H-type, and $\lambda_1 > \cdots > \lambda_m$ are all the eigenvalues of H. Let W_i be the eigenspace for the eigenvalue λ_i. Let $v \in W_1$, then Ev is an eigenvector with eigenvalue $\lambda_1 + 2$ and hence $Ev = 0$. By Lemma 2.1.8,

$$EF^r v = r(\lambda_1 - r + 1)F^{r-1}v.$$

This shows that the vectors $F^r v$ span a submodule of W. Since W is irreducible, the submodule spanned by these vectors is all of W. We have

$$HF^r v = (\lambda_1 - 2r)F^r v.$$

Since V is of finite H-type, there exists a natural number j such that $F^j v \neq 0$ and $F^{j+1}v = 0$. Let

$$v_i := F^i v,\ i = 0, 1, \cdots, j.$$

These vectors are linearly independent since they correspond to different eigenvalues of H, and they span all of W; i.e. they are a basis of W. By Lemma 2.1.8, $0 = EF^{j+1}v - F^{j+1}Ev = (j+1)(\lambda_1 - j)F^j v$. Note that $F^j v \neq 0$, so $\lambda_1 = j$. We sum up

$$Hv_i = (j - 2i)v_i$$

$$Fv_i = v_{i+1}$$

$$Ev_i = i(j - i + 1)v_{i-1}.$$

These equations completely determine the representation on the irreducible $sl(2)$-module.

Now let's get back to consider a representation on a possibly infinite dimensional $sl(2)$-module W of finite H-type

$$W = W_1 \oplus \cdots \oplus W_m,$$

where W_i is the eigenspace for the eigenvector λ_i.

Definition 2.1.11. *We call an element homogeneous if it belongs to one of the summands in the above decomposition of W. Moreover, we call an element $v \in W$ primitive if it is homogeneous and satisfies*

$$Fv = 0.$$

Repeating a similar proof given above, which we only used the finite H-type property, we see that eventually $E^l v = 0$ if v is primitive and that the cyclic submodule generated by v is finite dimensional and that

$$Hv = kv$$

for some non-negative integer k, where $k + 1$ is the dimension of the submodule generated by v since $v, Ev, \cdots, E^k v$ is a basis. To sum up, the eigenvalues of H are all integers. Hence by relabeling, we may decompose

$$W = \bigoplus W_k,\ H = k \cdot \mathrm{id}\ \mathrm{on} W_k.$$

Theorem 2.1.12. *The map*

$$E^k : W_{-k} \longrightarrow W_k$$

is bijective.

Proof. By the discussion above, we know the map is bijective if W is an irreducible. Therefore, this is also true for cyclic W since it is just a direct sum of finitely many irreducibles.

In general, let $v \in W_k$, then the map is bijective on the submodule generated by v. So there exists $u \in W_{-k}$ such that $E^k u = v$. This proves the surjectivity. Suppose $w \in W_{-k}$ such that $E^k w = 0$. Consider the submodule generated by w, then we get $w = 0$. This proves the injectivity. Thus, $E^k : W_{-k} \to W_k$ is a bijection in general. □

Theorem 2.1.13. *Let $v \in W_{-r}$ where $r \geq 0$, then v is primitive if and only if $E^{r+1}v = 0$.*

Proof. The discussion about the irreducible submodules above implies the necessity.

For the proof of the sufficiency, we first claim that v can be written as $\sum E^r v_r$ with v_r being primitive. Indeed, we have already seen that any element in an irreducible can be written as such form. Moreover, the submodule generated by v is a direct sum of finitely many irreducibles. Thus, v can be written as such form.

In particular, $v = u + Eu'$ where u and u' are both primitive. Apply E^{r+1} on the both sides of the equation and note $E^{r+1}u = 0$ since u is primitive, we have

$$E^{r+2}u' = 0.$$

Note that $u'\mathcal{W}_{-r-2}$, then by Theorem 2.1.12 we have $u' = 0$. Therefore, $v = u$ is primitive. □

Theorem 2.1.14. *Every $v \in W_p$ can be written as a finite sum*

$$v = \sum_{r \geq max\{0,p\}} E^r v_r$$

where all v_r are primitive. Moreover, there exists a non-commutative polynomial $\Psi_r(E, F)$ such that

$$v_r = \Psi_r(E, F)v.$$

Thus, the decomposition is unique.

Proof. In the proof of the last theorem, we have already shown that v can be written as the summation of $E^r v_r$ with v_r being primitive. By last theorem, $E^r v_r$ will be trivial for $r < \max\{0, p\}$. This is why the summation is from $\max\{0, p\}$.

We next find the non-commutative polynomial $\Psi_r(E, F)$. Write the summation explicitly

$$v = E^l v_l + E^{l-1} v_{l-1} + \cdots$$

Since the summation is from $\max\{0, p\}$, $l \geq p$. Apply E^{l-p} to the both sides of the equation above, all the terms disappear except the first one on the right hand side, i.e., $E^{l-p}v = E^{2l-p}v_l$.

We claim that if $u \in W_{-j}$ is a primitive element, then $F^k E^k u = Cu$ where C is a nonzero constant. Indeed,

$$\begin{aligned} F^k E^k u &= F^{k-1}(C_k E^{k-1} + E^k F)u \\ &= C_k F^{k-1} E^{k-1} u \\ &= \cdots \\ &= C_k C_{k-1} \cdots C_1 u \end{aligned}$$

Where C_i are the nonzero elements such that $[F, E^i] = C_i E^{i-1}$.

Therefore, $F^{2l-p}E^{l-p}v = F^{2l-p}E^{2l-p}v_l = Cv_l$, i.e., $v_l = \frac{1}{C}F^{2l-p}E^{l-p}v$. So we find the non-commutative polynomial Ψ_l such that the leading primitive term $v_l = \Psi_l v$. Then

$$v - \Psi_l v = E^{l-1}v_{l-1} + \cdots .$$

Repeating the process above, we will find the polynomial for v_{l-1}. Continue to do this, we find all the required polynomials $\Psi_r(E, F)$. □

2.2 Hodge Theory on Odd Dimensional Symplectic Manifold

We discuss Hodge theory on odd dimensional symplectic Manifolds in this section. As mentioned in the introduction, we will study basic forms in the odd dimensional case. At each point of an odd dimensional symplectic manifold the tangent space is just an odd dimensional symplectic vector space, so all the results we derive on the linear space in the last section can be applied correspondingly on the manifold.

2.2.1 Star Operator on Odd Dimensional Symplectic Manifolds

Suppose that (M, ω, Ω) is a $(2n+1)$-dimensional symplectic manifold, then by Theorem 1.2.3, locally we have coordinates $x_1, y_1, \cdots, x_n, y_n, z$ such that

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$$

and the Reeb vector field is $R = \frac{\partial}{\partial z}$. So locally any basic form is a summation of $f(x_1, y_1, \cdots, x_n, y_n)\alpha_1 \wedge \cdots \wedge \alpha_n$ type forms, where $\alpha_i \in \{1, dx_i, dy_i, dx_i \wedge dy_i\}$. By Lemma 2.1.3, we have

$$\star(f(x_1, y_1, \cdots, x_n, y_n)\alpha_1 \wedge \cdots \wedge \alpha_n) = f(x_1, y_1, \cdots, x_n, y_n) \star_n \alpha_n \wedge \cdots \wedge \star_1 \alpha_1.$$

Note that $\star_i 1 = dx_i \wedge dy_i$, $\star_i dx_i = dx_i$, $\star_i dy_i = dy_i$ and $\star_i(dx_i \wedge dy_i) = 1$. It is now clear that $\star$ is a map from $\Omega^p_{\text{bas}}(M)$ to $\Omega^{2n-p}_{\text{bas}}(M)$, and $\star^2 = \text{id}$. We summarize this as the following theorem.

Theorem 2.2.1. *Suppose that (M, ω, Ω) is a $(2n+1)$-dimensional symplectic manifold, then there is a star operator on basic forms*

$$\star : \Omega^p_{bas}(M) \to \Omega^{2n-p}_{bas}(M),$$

for $p = 0, 1, \cdots, 2n$. Moreover, $\star^2 = id$ and hence $\star$ is an isomorphism for each p.

2.2.2 More Hodge Identities

There is one operator that belongs to manifolds exclusively, the exterior differential operator d. In subsection 2.1.2 we already derive some Hodge identities on odd dimensional symplectic vector space, which is still true on odd dimensional symplectic manifolds. This subsection gives more Hodge identities which involve d.

Definition 2.2.2. *Suppose that (M, ω, Ω) is a $(2n+1)$-dimensional symplectic manifold. Then the transpose operator δ of d is defined by:*

$$\delta\alpha = (-1)^p \star d \star \alpha$$

where $\alpha \in \Omega^p_{bas}(M)$, $p = 0, 1, \cdots, 2n$. It is clear that $\delta^2 = 0$.

Definition 2.2.3. *We call a basic form α on M harmonic if it is closed and coclosed, i.e. $d\alpha = 0$ and $\delta\alpha = 0$.*

Theorem 2.2.4. *For $\alpha \in \Omega_{bas}(M)$, $[E, d]\alpha = 0$ and $[F, \delta]\alpha = 0$; $[d, F]\alpha = \delta\alpha$ and $[E, \delta]\alpha = -d\alpha$.*

Proof. By Theorem 1.2.3, locally we have Darboux coordinates $x_1, y_1, \cdots, x_n, y_n, z$ such that $\omega = \sum dx_i \wedge dy_i$ and $R = \frac{\partial}{\partial z}$. Therefore, locally in the coordinate chart basic forms are differential forms of variables $x_1, y_1, \cdots, x_n, y_n$. Moreover, all the four operators E, F, d and δ are defined either pointwise or locally. So all the four formulas are true as in the ordinary symplectic Hodge theory. □

Theorem 2.2.5. *For $\alpha \in \Omega_{bas}(M)$, $[d, \delta]\alpha = 0$.*

Proof. By Theorem 2.2.4, $\delta = [d, F]$. Therefore,

$$[d, \delta] = [d, [d, F]] = d[d, F] + [d, F]d = ddF - dFd + dFd - Fdd = 0$$

□

Remark 2.2.6. *By Theorem 2.2.5, the kernel of δ is a subcomplex of the basic deRham complex. For convenience, we use the notation $\Omega_{har}(M)$ to denote the kernel of δ.*

Theorem 2.2.7. *Suppose that $\alpha \in \Omega_{bas}(M)$ is harmonic, then $E\alpha$ and $F\alpha$ are both harmonic. Therefore, all the harmonic forms consist of an $sl(2)$-submodule of $\Omega_{bas}(M)$.*

Proof. By Theorem 2.2.4, $dE\alpha = Ed\alpha = 0$; $\delta E\alpha = E\delta\alpha + d\alpha = 0$; $dF\alpha = Fd\alpha + \delta\alpha = 0$; $\delta F\alpha = F\delta\alpha = 0$. □

Corollary 2.2.8. *Suppose that M is a $(2n+1)$-dimensional symplectic manifold, then there are two bijective maps*

$$E^k : \Omega^{n-k}_{bas}(M) \to \Omega^{n+k}_{bas}(M),$$

$$E^k : \Omega^{n-k}_{har}(M) \to \Omega^{n+k}_{har}(M)$$

for $k = 0, 1, \cdots, n$.

Proof. Since both $\Omega_{\text{bas}}(M)$ and $\Omega_{\text{har}}(M)$ are both $sl(2)$-modules of finite H-type, it is clear that the maps are bijective by Theorem 2.1.12. □

We conclude this section with the the following lemma, which will be extremely useful in the next chapter. The proof can be found in [We].

Lemma 2.2.9 (Weil Identity). *For all $r \leq n-p$ and the primitive element $\alpha \in \Omega^p_{bas}(M)$,*

$$\star E^r\alpha = (-1)^{\frac{p(p+1)}{2}} \frac{r!}{(n-p-r)!} E^{n-p-r}\alpha.$$

For $r > n-p$, $E^r\alpha = 0$ since α is primitive, and hence $\star E^r\alpha = 0$. In particular, $\star E^r\alpha = C \cdot E^{n-p-r}\alpha$, where C is some constant.

Chapter 3

Mathieu's Theorem and the dδ-Lemma

We will prove Mathieu's theorem and the $d\delta$-Lemma. The first section describe a basic version of the strong Lefschetz property, which will be used in the statement of Mathieu's theorem and the $d\delta$-Lemma. We discuss primitivity in the second section, which is an important tool for the proof. The third and fourth sections are the proofs of Mathieu's theorem and the $d\delta$-Lemma respectively. We conclude the chapter with several interesting examples in the fifth section.

3.1 The Strong Lefschetz Property

In this section, we introduce the strong Lefschetz property for basic cohomology on odd dimensional symplectic manifolds.

Definition 3.1.1. *A $(2n+1)$-dimensional symplectic manifold (M, ω, Ω) has the strong Lefschetz property if*

$$E^k : H^{n-k}_{bas}(M) \to H^{n+k}_{bas}(M)$$

is surjective for $k = 0, 1, \cdots, n$, where $H_{bas}(M) = H(\Omega_{bas}(M), d)$ is the basic cohomology of M.

Lemma 3.1.2. *Suppose M is a compact connected $(2n+1)$-dimensional symplectic manifold and there exists a connection 1-form λ. Then ω^k represents a non-trivial element in $H^{2k}_{bas}(M)$ for $k = 0, 1, \cdots, n$.*

Proof. Suppose ω^k is a zero element in $H^{2k}_{\mathrm{bas}}(M)$, then there exists a basic $(2k-1)$-form γ such that

$$\omega^k = d\gamma$$

Therefore, $\omega^n = d\gamma \wedge \omega^{n-k} = d(\gamma \wedge \omega^{n-k})$. Then

$$\int_M \Omega = \int_M \lambda \wedge d(\gamma \wedge \frac{\omega^{n-k}}{n!}) = 0.$$

The last equality holds since $\lambda \wedge d(\gamma \wedge \frac{\omega^{n-k}}{n!} = -d(\lambda \wedge \gamma \wedge \frac{\omega^{n-k}}{n!}$. Hence, this leads to a contradiction. □

Lemma 3.1.3. *Suppose that M is a compact connected $(2n+1)$-dimensional symplectic manifold and that there exists a connection 1-form λ. Suppose that*

$$E^n : H^0_{bas}(M) \to H^{2n}_{bas}(M)$$

is surjective. Then the map above is actually an isomorphism. Moreover, a closed basic $2n$-form β represents a zero element in $H^{2n}_{bas}(M)$ if and only if

$$\int_M \lambda \wedge \beta = 0.$$

Proof. It it obvious that $H^0_{\mathrm{bas}}(M) \cong \mathbb{R}$. By lemma 3.1.2, we know ω^n is not a zero element in $H^{2n}_{\mathrm{bas}}(M)$. Therefore, the map $E^n : H^0_{\mathrm{bas}}(M) \to H^{2n}_{\mathrm{bas}}(M)$ is injective, and hence an isomorphism. Note that there is another linear map

$$H^{2n}_{\mathrm{bas}}(M) \to \mathbb{R}$$

defined by $\gamma \mapsto \int_M \lambda \wedge \gamma$, where λ is the connection 1-form. It is an isomorphic linear map because it maps the non-zero element $\frac{\omega^n}{n!}$ to a non-zero element $\int_M \Omega$. This completes the proof. ☐

Remark 3.1.4. *We impose the surjectivity condition in the lemma above, which is always satisfied if the manifold has the strong Lefschetz property. However, we will see some examples that do not have this property.*

3.2 Primitivity

We give the definition of primitive in the Definition 2.1.11. Primitive elements will play a crucial role in the proof of Mathieu's theorem and the $d\delta$-Lemma. This section will discuss primitivity.

In this section we *always* assume that (M, ω, Ω) is a $(2n+1)$-dimensional symplectic manifold that has the strong Lefschetz property.

Lemma 3.2.1. *Let α be in $\Omega^p_{bas}(M)$. Then α can be written uniquely as a sum*

$$\alpha = \sum_{r \geq (p-n)^+} E^r \alpha_r$$

with $\alpha_r \in \Omega^{p-2r}_{bas}(M)$ primitive, where $(p-n)^+ = max\{p-n, 0\}$. Moreover, there are non-commutative polynomials $\Phi_r(E, F)$ such that for every $\alpha \in \Omega^p_{bas}(M)$,

$$\alpha_r = \Phi_r(E, F)\alpha.$$

Therefore, if α is harmonic then all the α_r are harmonic.

Proof. This is just the manifold version of Lemma 2.1.14. ☐

Lemma 3.2.2. *If the harmonic form α above is exact, then the α_r's are exact.*

Proof. We need to use the strong Lefschetz property here. By Lemma 3.2.1,

$$\alpha = \sum_{r=(p-n)^+}^{m} E^r \alpha_r$$

where $p - 2m = 0$ or 1, and $(p-n)^+ = \max\{p-n, 0\}$. Thus $\alpha_{m+1} = 0$ is exact.

Let's assume by induction that α_r is exact for $r > k$ and conclude that α_k is exact. By the induction hypothesis

$$\alpha' = \alpha - \sum_{r>k} E^r \alpha_r = \sum_{r \leq k} E^r \alpha_r$$

is exact. Applying E^{n-p+k}, we get the identity

$$E^{n-p+k}\alpha' = E^{n-(p-2k)}\alpha_k + \sum_{r<k} E^{n-(p-k-r)}\alpha_r$$

and since α_r is primitive and of degree $(p-2r)$, all the terms in the summation are zero. Thus

$$E^{n-p+k}\alpha' = E^{n-(p-2k)}\alpha_k$$

The left side of the above equation is exact and α_k is closed; therefore by the strong Lefschetz property α_k is exact. $\square$

Corollary 3.2.3. *If $\alpha \in \Omega^p_{bas}(M)$ is harmonic and exact, it is also co-exact.*

Proof. By the last lemma, α_r's in

$$\alpha = \sum E^r \alpha_r$$

are exact. By Weil's identity (Lemma 2.2.9),

$$\star E^r \alpha_r = C \cdot E^{n-p+r}\alpha_r,$$

where C is some constant. Since α_r's are exact $\star\alpha$ is exact, i.e. α is co-exact. $\square$

Lemma 3.2.4. *If $\tau \in \Omega^k_{bas}(M)$ is primitive, then $d\tau = \alpha_0 + \omega \wedge \alpha_1$, where α_0 and α_1 are primitive.*

Proof. τ primitive $\Leftrightarrow \omega^{n-k+1}\tau = 0 \Rightarrow \omega^{n-k+1}d\tau = 0$. Let

$$d\tau = \alpha_0 + \omega\alpha_1 + \omega^2\alpha_2 + \cdots$$

with $\alpha_i \in \Omega_{\text{bas}}^{k+1-2i}(M)$ primitive. Then

$$\omega^{n-k+1} d\tau = \sum_i \omega^{n-k+1+i} \alpha_i = 0$$

By the uniqueness of decomposition, we have

$$\omega^{n-k+1+i} \alpha_i = 0$$

But α_i is of degree $k+1-2i$; so $\alpha_i = 0$ if and only if

$$\omega^{n-(k+1-2i)} \alpha_i = 0$$

However,

$$\omega^{n-(k+1-2i)} \alpha_i = \omega^{n-k+1+i+(i-2)} \alpha_i$$

and this is zero for $i \geq 2$. □

Suppose $\tau \in \Omega_{\text{bas}}^p(M)$ has the property that $d\tau$ is harmonic. Let

$$\tau = \sum_{r=(p-n)^+}^{m} \omega^r \alpha_r$$

α_r primitive. By Lemma 3.2.4

$$d\alpha_r = \beta_r + \omega \beta_r'$$

where β_r and β_r' are primitive.

Lemma 3.2.5. *β_r and β_r' are both exact.*

Proof. As discussed above

$$d\tau = \sum_{r=(p-n)^+}^{m} \omega^r (\beta_r + \beta_{r-1}')$$

where by Lemma 3.2.2, $\beta_r + \beta_{r-1}'$ is exact for all r. In particular, $\beta_{m+1} = 0$, so β_m' is exact. Hence $\beta_m = d\alpha_m - \omega\beta'_m$ is exact. Therefore, u'_{m-1} is exact... Finally, all the β_r and β'_r are exact. □

Corollary 3.2.6. *Each summand $\omega^r d\alpha_r$ in the sum*

$$d\tau = \sum \omega^r d\alpha_r$$

is co-exact. In particular, $d\tau$ is co-exact.

Proof. $\omega^r d\alpha_r = \omega^r \beta_r + \omega^{r+1}\beta'_r$, so by Lemma 2.2.9

$$\star(\omega^r d\alpha_r) = C_1\omega^{n-p+r-1}\beta_r + C_2\omega^{n-p+r}\beta'_r$$

where C_1, C_2 are some constants. Thus, $\star(\omega^r d\alpha_r)$ is exact, i.e. $\omega^r d\alpha_r$ is co-exact. □

3.3 Mathieu's Theorem

Mathieu's theorem in the ordinary symplectic Hodge theory states that every cohomology class has a harmonic representative if and only if the manifold has the strong Lefschetz property. The $d\delta$-Lemma states that if the symplectic manifold has the strong Lefschetz property, then for any harmonic form α, if it is exact, it is also co-exact (δ-exact); moreover, there exists a differential form β such that

$$\alpha = d\delta\beta.$$

Now on an odd-dimensional symplectic manifold M, we shall try to obtain similar results except that in our case we study basic forms instead. As mentioned right after the corollary 2.2.5, $(\ker\delta, d)$ is a subcomplex of the basic deRham complex $(\Omega_{\text{bas}}(M), d)$. Recall that we use the following notation

$$\Omega_{\text{har}}(M) = \{\alpha \in \Omega_{\text{bas}}(M) \mid \delta\alpha = 0\}$$

to denote this subcomplex and we define the harmonic cohomology to be

$$H_{\text{har}}(M) := \frac{\ker d \cap \Omega_\delta}{d(\Omega_\delta)}.$$

Theorem 3.3.1 (Mathieu's Theorem). *Suppose M is a compact $(2n+1)$-dimensional symplectic manifold. Then (M, ω, Ω) has the strong Lefschetz property if and only if every basic cohomology class has a harmonic representative. Equivalently, M has the strong Lefschetz property if and only if the natural map*

$$H^r_{har}(M) \to H^r_{bas}(M)$$

is surjective for all $r = 0, 1, \cdots, 2n$.

Remark 3.3.2. *The 'if' part is straightforward. Indeed, suppose that $H^r_{har} \to H^r_{bas}$ is surjective for all r, then for any closed basic $(n+l)$-form $\alpha \in \Omega^{n+l}_{bas}(M)$, there exists $\beta \in \Omega^{n+l}_{har}(M)$ such that $[\alpha] = [\beta]$. By Corollary 2.2.8, there exists a harmonic $(n-l)$-form γ such that $E^l\gamma = \beta$, and hence $E^l[\gamma] = [\alpha]$. This completes the proof of 'if' part.*

Proof. We only need to focus on the 'only if' part now. Assume that M has the strong Lefschetz property. We first prove $H^r_{\text{har}}(M) \to H^r_{\text{bas}}(M)$ is surjective for $0 \le r \le n$ by induction. It is trivially true for $r = 0$. For $r = 1$, let $[\alpha] \in H^1_{\text{bas}}(M)$, where α is a closed basic 1-form. Note α is automatically a primitive form. Then by Theorem 2.2.4,

$$\delta\alpha = [d, F]\alpha = 0,$$

i.e. α is harmonic. So it is also true for $r = 1$.

Now suppose $H^r_{\text{har}}(M) \to H^r_{\text{bas}}(M)$ is surjective for all $r < n-k$. We shall prove it is also true for $r = n-k$. Let $\alpha \in \Omega^{n-k}_{\text{bas}}$ and $d\alpha = 0$. By the strong Lefschetz property $E^{k+1}\alpha \in H^{n+k+2}_{\text{bas}}(M)$ is in the image of the map

$$E^{k+2} : H^{n-k-2}_{\text{bas}}(M) \to H^{n+k+2}_{\text{bas}}(M).$$

Therefore, there exist a closed basic $(n-k-2)$-form $\beta \in \Omega^{n-k-2}_{\text{bas}}(M)$ and basic $(n+k+1)$-form $\gamma \in \Omega^{n+k+1}_{\text{bas}}(M)$ such that

$$E^{k+1}\alpha = E^{k+2}\beta + d\gamma$$

Thus,

$$E^{k+1}(\alpha - E\beta) = d\gamma$$

By Corollary 2.2.8, there exists a basic closed $(n-k-1)$-form $\theta \in \Omega^{n-k-1}_{\text{bas}}$ such that

$$\gamma = E^{k+1}\theta$$

Then

$$E^{k+1}(\alpha - E\beta - d\theta) = 0,$$

i.e., $\alpha - E\beta - d\theta$ is primitive. Note that it is closed as well, so

$$\delta(\alpha - E\beta - d\theta) = [d, F](\alpha - E\beta - d\theta) = 0.$$

Therefore, $\alpha - E\beta - d\theta$ is harmonic. Now we use the induction hypothesis: $[\beta]$ has a harmonic representation β'. So $[E\beta]$ has a harmonic representative $E\beta'$. Thus,

$$[\alpha] = [(\alpha - E\beta - d\theta) + E\beta'].$$

Finally, $(\alpha - E\beta - d\theta) + E\beta'$ is a harmonic representative of $[\alpha]$.

So far we have proved that $H^r_{\text{har}} \to H^r_{\text{bas}}$ is surjective for $0 \le r \le n$. For $r = n + l$, let $\alpha \in \Omega^{n+l}_{\text{bas}}(M)$ be a closed basic form. By the strong Lefschetz property, there exists a closed basic form $\beta \in \Omega^{n-l}_{\text{bas}}$ such that

$$\alpha = E^k\beta + d\eta$$

for some $\eta \in \Omega^{n+l-1}_{\text{bas}}$. Since we have already proven that $[\beta]$ has a harmonic representative β', $E^k\beta'$ is a harmonic representative of $[\alpha]$. This completes the proof of

Mathieu's theorem. $\square$

3.4 The dδ-Lemma

Theorem 3.4.1 ($d\delta$-Lemma). *Suppose (M, ω, Ω) is a compact $(2n+1)$-dimensional symplectic manifold with a connection 1-form λ and M has the strong Lefschetz property. Then for any exact harmonic form $\alpha \in \Omega^r_{bas}(M)$ (i.e. $\alpha = d\beta$ for some $\beta \in \Omega^{r-1}_{bas}(M)$ and $\delta\alpha = 0$), there exists $\gamma \in \Omega^r_{bas}(M)$ such that*

$$\alpha = d\delta\gamma.$$

In particular, this implies that the map

$$H_{har}(M) \to H_{bas}(M)$$

is bijective.

Proof. Once again we will use induction to prove the $d\delta$-lemma. First we need verify the initial steps. The case for $r = 0$ is trivial. For $r = 1$, let $\alpha \in \Omega^1_{\mathrm{har}}(M)$ and $f \in \Omega^0_{\mathrm{bas}}(M)$ such that $\alpha = df$. We assume that $\int_M f\Omega = 0$, otherwise we can add some constant to the basic function f. Thus,

$$\int_M \lambda \wedge (f\frac{\omega^n}{n!}) = \int_M f\Omega = 0.$$

By Lemma 3.1.3, the linear map $H^{2n}_{\mathrm{bas}} \to \mathbb{R}$ defined by $\tau \mapsto \int_M \lambda \wedge \tau$ is bijective. Therefore, $f\frac{\omega^n}{n!}$ represents a zero element in $H^{2n}_{\mathrm{bas}}(M)$, i.e., there exists a basic form θ such that

$$f\frac{\omega^n}{n!} = d\theta$$

Now apply $\star$ to both sides of the above equation, we get

$$f = \star d\theta = \star d \star (\star\theta).$$

Equivalently,

$$f = \delta(-\star\theta).$$

Let $\gamma = -\star\theta \in \Omega^1_{\text{bas}}(M)$, then

$$\beta = df = d\delta\gamma$$

This completes the verification of the initial steps.

Now suppose the $d\delta$-Lemma is true for $r < p$, we will prove that it is also true for $r = p$. Let $\alpha = d\beta$ for some $\beta \in \Omega^{p-1}_{\text{bas}}(M)$ and $\delta\alpha = 0$. By Lemma 3.2.1, we have the following decomposition

$$\beta = \sum_{l \geq (n-p)_+} E^l \beta_l$$

where β_l's are primitive (i.e., $F\beta_l = 0$). By Lemma 3.2.4, $d\beta_l = \gamma_l + \omega \wedge \gamma_l'$, γ_l and γ_l' both being primitive. Note that $d\beta (= \alpha)$ is harmonic, so γ_l and γ_l' are both exact by Lemma 3.2.2. So $E^l d\beta_l = \omega^l \wedge \gamma_l + \omega^{l+1} \wedge \gamma_l'$ is both exact and co-exact. Therefore, we only need to prove the $d\delta$-lemma for basic $(p-1)$-forms β, of the form $\beta = \omega^l v$, where v is primitive of degree $p - 2l - 1$. So $dv = u_1 + \omega u_2$, u_1 and u_2 both being primitive and exact. By Lemma 2.2.9,

$$\star(\omega^l v) = C\omega^{n-p+l+1} v$$

where C is some constant. Hence,

$$d\star(\omega^l v) = C(\omega^{n-p+l+1} u_1 + \omega^{n-p+l+2} u_2)$$

Now we apply $\star$ on both sides of the above equation and use the Weil identity again

$$\delta(\omega^l v) = C_1 \omega^{l-1} u_1 + C_2 \omega^l u_2$$

Therefore, $\delta(\omega^l v) \in \Omega^{p-2}_{\text{bas}}(M)$ is both exact and co-exact. By the induction hypothesis, there exists $\theta \in \Omega^{p-2}_{\text{bas}}$ such that

$$\delta(\omega^l v) = \delta d\theta$$

Let $\gamma = \omega^l v - d\theta$. Then $\alpha = d\gamma$ and $\delta\gamma = 0$. Since γ is co-closed, it has a harmonic representative η of the δ-cohomology class $[\gamma]_\delta$ by Mathieu's theorem; i.e., $\gamma = \eta + \delta\mu$ for some $\mu \in \Omega^p_{\text{bas}}(M)$ and $\delta\eta = 0$ and $d\eta = 0$. Thus

$$\alpha = d\gamma = d\eta + d\delta\mu = d\delta\mu$$

This completes the proof of the induction. □

We have already introduced two types of cohomologies on odd-dimenisonal symplectic manifolds, the basic cohomology H_{bas} and the harmonic cohomology H_{har}. The following is a third:

Definition 3.4.2. *The symplectic cohomology H_{sympl} on an odd dimensional symplectic manifold M is defined by:*

$$H_{sympl}(M) = \frac{\ker d \cap \ker \delta}{d(\Omega_{bas}(M)) \cap \ker \delta}$$

Recall that the harmonic cohomology is defined as

$$H_{\text{har}} = \frac{\ker d \cap \ker \delta}{d(\Omega_\delta)}$$

and observe that $d(\Omega_\delta) \subset d(\Omega_{\text{bas}}) \cap \ker \delta$, so there is a natural surjective map

$$H_{\text{har}} \longrightarrow H_{\text{sympl}}.$$

Also note that H_{sympl} is a subgroup of H_{bas}, we have

$$H_{\text{har}} \longrightarrow H_{\text{sympl}} \mapsto H_{\text{bas}}$$

By the $d\delta$-lemma, if the odd dimensional symplectic manifold has the strong Lefschetz property, the composition of the two maps above is an isomorphism. Therefore, both maps are isomorphic, so we have the following theorem.

Theorem 3.4.3. *If an odd-dimensional symplectic manifold has the strong Lefschetz property, all three cohomologies H_{har}, H_{bas} and H_{sympl} are isomorphic.*

3.5 Examples

Example 3.5.1 (Hopf Fibration). *We start with the classical Hopf Fibration*

$$\pi : S^{2n-1} \longrightarrow \mathbb{CP}^{n-1}.$$

Regard S^{2n-1} as the unit sphere in $\mathbb{C}^n$, use the polar coordinates $(r_1, \theta_1, \cdots, r_n, \theta_n)$, and let ω_{FS} be the Fubini-Study symplectic 2-form on $\mathbb{CP}^{n-1}$, then

$$\pi^* \omega_{FS} = \sum_{i=1}^{n} d(r_i^2) \wedge d\theta_i,$$

which can serve as the symplectic 2-form on S^{2n-1}. We denote it by ω. Note that $\sum_{i=1}^{n} r_i^2 d\theta_i$ is a connection 1-form on the Hopf Fibration. The corresponding Reeb vector field is

$$R = \sum_{i=1}^{n} \frac{\partial}{\partial \theta_i}.$$

All of those make S^{2n-1} an odd dimensional symplectic manifold.

Since it is a circle bundle over the symplectic manifold $(\mathbb{CP}^{n-1}, \omega_{FS})$, the odd dimensional symplectic Hodge theory on S^{2n-1} is exactly the symplectic Hodge theory on $\mathbb{CP}^{n-1}$. Therefore, $H^{2k}_{bas}(S^{2n-1}) \cong H^{2k}(\mathbb{CP}^{n-1}) = \mathbb{R}$ for $k = 0, 1, \cdots, n-1$. Furthermore, we have already shown that $[\omega^k]$ is a non-trivial element in $H^{2k}_{bas}(S^{2n-1})$, so $[\omega^k]$ generates the $2k$-th basic cohomology group. And we conclude that this odd dimensional symplectic manifold has the strong Lefschetz property. By Mathieu's theorem and the $d\delta$-Lemma, the harmonic cohomology is isomorphic to the basic cohomology.

We will next present two examples in which H_{bas} is infinite dimensional but H_{har} is finite dimensional.

Example 3.5.2 (Contact Structure on T^3). *There is a classical contact* 1*-form on T^3: $\lambda = (\cos x)dy + (\sin x)dz$, where $x, y, z \in [0, 2\pi)$ are the local coordinates for the three S^1-components in T^3. The* 1*-form is well defined globally since both* cos *and* sin *functions are of period 2π. The contact* 1*-form's exterior differential is*

$$\omega = d\lambda = (-\sin x)dx \wedge dy + (\cos x)dx \wedge dz.$$

Therefore, the Reeb vector field is

$$R = (\cos x)\frac{\partial}{\partial y} + (\sin x)\frac{\partial}{\partial z}.$$

Firstly, let's try to find all the basic 0*-forms on T^3, i.e. these smooth functions on T^3 whose Lie derivatives over Reeb vector field vanishes everywhere. Note that the Reeb vector field R has no $\frac{\partial}{\partial x}$ term, so every orbit lies within the level set $x = a$ for some $a \in [0, 2\pi)$. Let $f(x, y, z)$ be a basic function, then $\mathcal{L}(R)f = 0$. We claim that f is just a function of variable x.*

Indeed, on any level set $x = x'$ such that $(\sin x' / \cos x')$ is an irrational number, the Reeb vector field is irrational, and thus the function f will be constant on the level set. Most level sets will be of this 'irrational' type except for a countable set of values of x'. By the continuum property of the function, f will be constant on all the level sets of x, i.e. it is just a function of variable x. It is now legitimate to write $f = f(x)$, and we conclude that

$$\Omega^0_{bas}(T^3) = \{f(x) |\ f \in C^\infty(T^3)\}.$$

There is a global base for $\Omega^1(T^3)$: dx, dy, dz. This enables us to write all the 1*-forms as a linear combination of dx, dy and dz.*

Let's now try to figure out what the Lie derivatives of dx, dy, dz over R are.

Recall the Cartan formula $\mathcal{L}(R) = \iota(R)d + d\iota(R)$. *Then*

$$\mathcal{L}(R)dx = 0,$$

$$\mathcal{L}(R)dy = d\iota(R)dy = d(\cos x) = (-\sin x)dx,$$

$$\mathcal{L}(R)dz = d\iota(R)dz = d(\sin x) = (\cos x)dx.$$

Suppose that a 1*-form* $fdx + gdy + hdz$ *is basic, where* f, g, h *are all smooth functions on* T^3. *Then* $\iota(R)(fdx + gdy + hdz) = 0$ *and* $\mathcal{L}(R)(fdx + gdy + hdz) = 0$, *which are equivalent to*

$$g \cos x + h \sin x = 0,$$

$$(\mathcal{L}(R)f - g \sin x + h \cos x)dx + (\mathcal{L}(R)g)dy + (\mathcal{L}(R)h)dz = 0,$$

The second equation above is equivalent to $\mathcal{L}(R)f - g\sin x + h\cos x = \mathcal{L}(R)g = \mathcal{L}(R)h = 0$. *Therefore,* g *and* h *are both basic functions, i.e.* $g = g(x)$ *and* $h = h(x)$. *Moreover, note that* $\mathcal{L}(R)f = g(x)\sin x - h(x)\cos x$, *so the Lie derivative of* f *over* R *is a function of* x. *We claim that* f *is also a basic function.*

Indeed, on any level set $x = x'$, $\mathcal{L}(R)f$ *equals* $g(x')\sin x' - h(x')\cos x'$, *which is a constant. We denote this constant by* $C(x')$. *Moreover, recall that* $\mathcal{L}(R)dy = d(\cos x)$, $\mathcal{L}(R)dz = d(\sin x)$, *so on the level set* $\mathcal{L}(R)dy = \mathcal{L}(R)dz = 0$. *In particular,* $\mathcal{L}(R)(dy \wedge dz) = 0$. *Therefore, on the level set* $x = x'$

$$(\mathcal{L}(R)f)dy \wedge dz = \mathcal{L}(R)(fdy \wedge dz) - f\mathcal{L}(R)(dy \wedge dz) = \mathcal{L}(R)(fdy \wedge dz).$$

Now apply the integral over the level set on both sides of the equation above and not that $\mathcal{L}(R)f = C(x')$ *is a constant, we have*

$$4\pi^2 C(x') = \int_{x=x'} (\mathcal{L}(R)f)dy \wedge dz = \int_{x=x'} \mathcal{L}(R)(fdy \wedge dz) = 0.$$

The right hand equality holds because the integrand is the Lie derivative of some top form on the level set. Therefore, $C(x') = 0$ *for any* x', *i.e.,* $\mathcal{L}(R)f = 0$ *and hence*

$f = f(x)$.

Note that we also know $g(x)\cos x + h(x)\sin x = 0$, *so* $g \equiv h \equiv 0$. *Thus we are able to find all the basic* 1*-forms:*

$$\Omega^1_{bas}(T^3) = \{f(x)dx|\ f \in C^\infty(S^1)\}.$$

Finally, we try to figure out what the basic 2*-forms look like. Suppose that* $f dy \wedge dz + g dz \wedge dx + h dx \wedge dy$ *is a basic* 2*-form. Then the basic* 2*-form must be annihilated by* R*:*

$$(-h\cos x + g\sin x)dx - f\sin x dy + f\cos x dz = 0,$$

which is equivalent to $f \equiv 0$ *and* $-h\cos x + g\sin x = 0$. *Thus, the* 2*-form can be simplified to be* $g dz \wedge dx + h dx \wedge dy$. *Note that* $\mathcal{L}(R)(dz \wedge dx) = 0$ *and* $\mathcal{L}(R)(dx \wedge dy) = 0$, *so the* R*-invariance of the basic* 2*-form is equivalent to:* $\mathcal{L}(R)g = \mathcal{L}(R)h = 0$. *Therefore,* $g = g(x)$ *and* $h = h(x)$. *Also note that* $-h(x)\cos x + g(x)\sin x = 0$ *is equivalent to* $h(x)dy - g(x)dz = F(x)(-\sin x dy + \cos x dz)$ *for some* $F \in C^\infty(S^1)$. *Finally we recall that* $dx \wedge (-\sin x dy + \cos x dz) = \omega$, *and conclude that*

$$\Omega^2_{bas}(T^3) = \{F(x)\omega|\ F \in C^\infty(S^1)\}.$$

Now it is clear that $H^0_{bas} \cong \mathbb{R}$, $H^1_{bas} \cong \mathbb{R}$, *and* $H^2_{bas} \cong C^\infty(S^1)$. *Therefore, the odd dimensional symplectic manifold* T^3 *does not have the strong Lefschetz Property. It can be easily checked that the basic* 0*- and* 1*-forms are all harmonic, but*

$$\Omega^2_{har}(T^3) = \{C\omega|\ C \in \mathbb{R}\}.$$

Thus, $H^2_{har} \cong \mathbb{R}$, *which is not isomorphic to* H^2_{bas} *as expected.*

Example 3.5.3. *Equip the two-torus* T^2 *with the standard symplectic structure* $\omega_0 = dx \wedge dy$, *where* x, y *are the local coordinates for the two* S^1*-components of* T^2. *Let's* *consider a symplectic linear transformation φ on T^2 (i.e. $\varphi \in SL(2,\mathbb{Z})$) defined by:*

$$\varphi(x,y) := (x+y, y).$$

We have discussed in Chapter 1 that the symplectic structure on T^2 naturally extends to an odd dimensional symplectic structure on the mapping torus of φ. Moreover, we are going to prove the following lemma.

Lemma 3.5.4. *Suppose (X, ω_0) is a symplectic manifold, and $\varphi : X \to X$ is a symplectic transformation, i.e. φ is a diffeomorphism and $\varphi^*\omega_0 = \omega_0$. We denote the mapping torus of φ by X_φ. Then there is an isomorphism between the differential complex of φ-invariant differential forms on X and the differential complex of basic differential forms on X_φ, i.e.*

$$(\Omega(X)^\varphi, d_X) \cong (\Omega_{bas}(X_\varphi), d_{X_\varphi}),$$

where d_X and d_{X_φ} are exterior differentials on X and X_φ respectively.

Proof. The essential point here is that the mapping torus X_φ is locally just the trivial odd dimensional symplectic manifold $X \times (-\varepsilon, \varepsilon)$. Therefore, every differential form on X locally bijectively corresponds to a basic form on X_φ. For this to work globally (i.e. local pieces can be glued), we only need the differential form on X be φ-invariant. So we establish a bijective map $F : \Omega(X)^\varphi \to \Omega_{\text{bas}}(X_\varphi)$.

Now once again by using that locally X_φ is trivially symplectic, it is clear that F is actually an isomorphism between the complexes $(\Omega(X)^\varphi, d_X)$ and $(\Omega_{\text{bas}}(X_\varphi), d_{X_\varphi})$, i.e. $F \circ d_X = d_{X_\varphi} \circ F$. ☐

Remark 3.5.5. *Not only we can define a star operator $\star_{X_\varphi}$ on the odd dimensional symplectic manifold X_φ, but we can also define a star operator $\star_X$ on the ordinary symplectic manifold X. By the definition of the star operators on odd dimensional symplectic manifolds and the way we construct odd dimensional symplectic structures on the mapping tori of symplectic transformations, it is clear that F preserves the star operator as well, i.e. $F \circ \star_X = \star_{X_\varphi} \circ F$. So the odd dimensional symplectic Hodge* *theory on the mapping tori X_φ is exactly the φ-invariant symplectic Hodge theory on X.*

Now we get back to the original T^2 case, and try to find all the φ-invariant forms. Suppose $f(x,y) \in \Omega^0(T^2)^\varphi$, then $f(x,y) = f(x+y,y)$ for any x, y. Given any irrational number y_0, $f(x,y_0) = f(x+y_0,y_0)$. Replace x by $x+y_0$ in the equation, we get $f(x+y_0,y_0) = f(x+2y_0,y_0)$. Generally, $f(x+ny_0,y_0)$'s are the same for all $n \in \mathbb{Z}$. Note that if y_0 is an irrational number, then $\{x+ny_0\}_{n=-\infty}^{\infty}$ is a dense subset in S^1. Therefore, $f(x,y)$ is just a function of the variable y for all the irrational number y. Finally, by continuum we know that it is true for all y, i.e.

$$\Omega^0(T^2)^\varphi = \{f(y)|\ f \in C^\infty(S^1)\}$$

Suppose $g(x,y)dx + h(x,y)dy \in \Omega^1(T^2)^\varphi$, then $g(x,y)dx + h(x,y)dy = g(x+y,y)dx + (g(x+y,y) + h(x+y,y))dy$, i.e. $g(x,y) = g(x+y,y)$ and $h(x,y) = g(x+y,y) + h(x+y,y)$. Therefore, g is a φ-invariant function and hence $g = g(y)$. Then the second equation becomes $h(x,y) = g(y) + h(x+y,y)$. Replacing x by $x+y$ in the equation, we get $h(x+y,y) = g(y) + h(x+2y,y)$. Combining these two equations, we have $h(x+2y,y) = 2g(y) + h(x,y)$. Repeating this process, we have $h(x+ny,y) = ng(y) + h(x,y)$, or $g(y) = \frac{1}{n}(h(x+ny,y) - h(x,y))$. Apply $n \to \infty$ on both sides and note that h is a bounded function on T^2, we get $g \equiv 0$. Moreover, plug $g(y) = 0$ into the equation, we have $h(x,y) = h(x+y,y)$. By the discussion of φ-invariant 0-form above, we know that h is just a function of variable y, i.e. $h = h(y)$. We conclude that

$$\Omega^1(T^2)^\varphi = \{h(y)dy|\ h \in C^\infty(S^1)\}$$

Suppose $g(x,y)dx \wedge dy$ is a φ-invariant 2-form, then $g(x,y)$ is an invariant 0-form, i.e. $g = g(y)$. So

$$\Omega^2(T^2)^\varphi = \{g(y)dx \wedge dy|\ g \in C^\infty(S^1)\}$$

Finally, we conclude that $H^0_\varphi(T^2) \cong \mathbb{R}$, $H^1_\varphi(T^2) \cong \mathbb{R}$, and $H^2_\varphi(T^2) \cong C^\infty(S^1)$. So it does not have the strong Lefschetz property. Note that all the invariant 0- and

1-forms are harmonic, but $\Omega^2_{har}(T^2)^{\varphi} = \{Cdx \wedge dy|\ C \in \mathbb{R}\}$. *Therefore, the second invariant harmonic cohomology group is* $\mathbb{R}$, *which is not isomorphic to* $H^2_{\varphi}(T^2)$ *as expected.*

More interestingly, the second invariant cohomology group is infinite dimensional while the second invariant harmonic cohomology group is finite dimensional.

Example 3.5.6 (Mapping Torus of the Cat Map). *In this example, we still consider a linear symplectic transformation on* T^2 *with standard symplectic structure* $\omega_0 = dx \wedge dy$. *The only difference is that we use the cat map* ϕ *as the symplectic transformation instead, i.e.* $\phi(x, y) = (2x + y, x + y)$. *First we state some of the well-known properties for the cat map:*

1. ϕ *has a unique hyperbolic fixed point (0,0), where hyperbolic means the eigenvalues of the linear transformation are real numbers, one greater than* 1 *and the other less than* 1.

2. ϕ *is topologically transitive, i.e. there exists a point whose orbit is dense.*

By property 2, it is clear that ϕ*-invariant functions are all constant functions, i.e.*

$$\Omega^0(T^2)^{\phi} = \{C|\ C \in \mathbb{R}\}$$

For the same reason, ϕ*-invariant* 2*-forms are of the form* $C\omega_0$.

$$\Omega^2(T^2)^{\phi} = \{C\omega_0|\ C \in \mathbb{R}\}$$

Now let's try to find all the ϕ*-invariant* 1*-forms on* T^2. *Note that the linear transformation* ϕ *has two eigenvalues* λ_1, λ_2 *where* $\lambda_1 > 1 > \lambda_2 > 0$, *and two corresponding eigenvectors* $v_1 = (a_1, b_1)$ *and* $v_2 = (a_2, b_2)$. *We let* $\alpha_1 = a_1 dx + b_1 dy$, $\alpha_2 = a_2 dx + b_2 dy$, *then* α_1 *and* α_2 *is a base of* $\Omega^1(T^2)$, *and* $\phi^*\alpha_i = \lambda_i \alpha_i$, $i = 1, 2$.

Suppose $\alpha = f\alpha_1 + g\alpha_2$ *is a* ϕ*-invariant* 1*-form, then* $(\phi^*)^n\alpha = \alpha$, *that is* $\lambda_1^n(f \circ$

$\phi^n)\alpha_1 + \lambda_2^n(g \circ \phi^n)\alpha_2 = f\alpha_1 + g\alpha_2.$, *i.e. for any* $n \in \mathbb{Z}$

$$f = \lambda_1^n(f \circ \phi^n)$$

$$g = \lambda_2^n(g \circ \phi^n)$$

Note that f *and* g *are smooth functions on the compact manifold* T^2*, so they are both bounded. Then let* $n \to -\infty$ *in the first equation and note that* $\lambda_1 > 1$*, we get* $f \equiv 0$*. Similarly, let* $n \to \infty$ *in the second equation and note that* $0 < \lambda_2 < 1$*, we get* $g \equiv 0$*. Therefore,*

$$\Omega^1(T^2)^\phi = 0$$

We conclude that $H^0_\phi(T^2) \cong \mathbb{R}$*,* $H^1_\phi(T^2) \cong 0$*, and* $H^2_\phi(T^2) = \mathbb{R}$*. So the strong Lefschetz property holds. On the other hand, it is clear that all the* ϕ*-invariant forms are harmonic, so the invariant harmonic cohomology is the same as the invariant cohomology as expected.*

Chapter 4

Group Actions on Odd Dimensional Symplectic Manifolds

4.1 Hamiltonian Actions

In this section we discuss group actions on the odd dimensional symplectic manifolds.

Suppose that G is a Lie Group and M is an odd dimensional symplectic manifold. An action of G on M requires that

1. G acts smoothly on M in the usual sense;
2. the action preserves the symplectic 2-form ω and the volume form Ω.

Hence, the action also preserves the Reeb vector field R. Furthermore, we can define a group action on M to be Hamiltonian as follows.

Definition 4.1.1. *A group action of G on the odd dimensional symplectic manifold M is Hamiltonian if there exists an equivariant map*

$$\Psi : M \to \mathfrak{g}^*$$

such that

$$d\langle \Psi, X \rangle = \iota(X_M)\omega$$

where $\mathfrak{g} = Lie(G)$, and X_M is the vector field generated by $X \in \mathfrak{g}$.

Example 4.1.2 (Hamiltonian Action). *There is an inclusion map $S^3 \hookrightarrow \mathbb{C}^2$, then we have complex coordinates z_1, z_2 such that $|z_1|^2 + |z_2|^2 = 1$ to represent the points on the unit sphere S^3. We can also use polar coordinates $r_1, \theta_1, r_2, \theta_2$, where $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$. On the unit sphere, $r_1^2 + r_2^2 = 1$. The volume form is*

$$\Omega = d(r_1^2) \wedge d\theta_1 \wedge d\theta_2$$

and there is a canonical contact 1-form

$$\lambda = r_1^2 d\theta_1 + r_2^2 d\theta_2.$$

Thus, the symplectic 2-form is

$$\omega = d(r_1^2) \wedge d\theta_1 + d(r_2^2) \wedge d\theta_2.$$

Note that there is a T^2-action on S^3 which rotates the angle coordinates θ_1 and θ_2. Accordingly, the moment map can be chosen to be

$$\Psi(r_1, \theta_1, r_2, \theta_2) = (r_1^2, r_2^2).$$

Example 4.1.3 (Non-Hamiltonian Action). *In the Example 3.5.3, we study the mapping torus of the symplectic transformation $\phi : T^2 \to T^2$, $(x, y) \mapsto (x+y, y)$. Consider an S^1-action on the mapping torus M_ϕ by rotating the angle coordinate x. Note that $\iota(\frac{\partial}{\partial x})(dx \wedge dy) = dy$, so the action is not Hamiltonian.*

We next prove a version of equivariant Darboux theorem for odd dimensional symplectic G-manifolds. This is important in the proof of the local convexity theorem later. Note that in the Equivariant Darboux Theorem we do *not* require either the group G to be abelian or the action to be Hamiltonian.

Theorem 4.1.4. *Suppose that G is a compact connected Lie group, M is an odd dimensional symplectic G-manifold, and N is a G-invariant submanifold such that*

R is nowhere tangent to N, i.e., $R(n) \notin T_nN$ for all $n \in N$. Then there exist a G-invariant symplectic submanifold W of codimension 1 (in the even dimensional sense) that contains N and a G-invariant neighborhood U of N, and an equivariant diffeomorphism

$$\varphi : U \to W \times (-\varepsilon, \varepsilon)$$

such that $d\varphi(R) = \frac{\partial}{\partial t}$, where t is the coordinate for the interval $(-\varepsilon, \varepsilon)$ and ε is a small enough positive number. In addition, the action on $W \times (-\varepsilon, \varepsilon)$ is defined as

$$a \cdot (w, t) = (a \cdot w, t)$$

for any $a \in G$, $w \in W$ and $t \in (-\varepsilon, \varepsilon)$.

Proof. Since N is transverse to the Reeb vector flows, we can find a Riemannian metric on M, such that the Reeb vector field is perpendicular to N. By averaging the chosen metric by the compact group G we can further get a G-invariant metric g, with which the Reeb vector field is still perpendicular to N. Now with this metric, we have the following splitting

$$TM|_N = TN \oplus TN^{\perp}.$$

Then R is tangent to $TN^{\perp}$, i.e., R is a non-vanishing section of the normal bundle $TN^{\perp} \to N$. Now let $E \to N$ be the subbundle whose fibers are defined by: $E_n = R_n^{\perp} \cap T_nN^{\perp}$. Note that both R and the metric g are G-invariant, hence is the subbundle E.

By the equivariant version of the tubular neighborhood theorem, we know that a G-invariant neighborhood U of N is equivariantly diffeomorphic to a neighborhood of the zero section of the normal bundle $TN^{\perp}$, where it identifies N as the zero section of the normal bundle. Therefore, the subbundle E is diffeomorphic to a G-invariant submanifold W of codimension 1 that contains N, since E has co-rank 1 and contains the zero section.

Now we are ready to construct the G-equivariant map $\varphi : U \to W \times (-\varepsilon, \varepsilon)$.

First, for any $w \in W \subset U$, $\varphi : w \mapsto (w, 0)$. Second, by the definition of tubular neighbourhood for any $m \notin W$, there exists a unique point $w \in W$, and $s \in (-\varepsilon, \varepsilon)$, such that $\phi_R^s(w) = m$, where ϕ_R^t is the local diffeomorphism generated by the Reeb vector field. Now we can define $\varphi : m \mapsto (w, s)$. Since both R and W are G-invariant, φ is really an G-equivariant map. From the construction above it is clear that $d\varphi(R) = \frac{\partial}{\partial t}$. $\square$

Remark 4.1.5. *If we apply the above theorem to the case that N is a fixed point, in which case the tangency condition is always satisfied, we get an equivariant version of Darboux charts for odd dimensional symplectic G-manifolds, i.e., locally we have coordinates $x_1, y_1, \cdots, x_n, y_n, z$ such that*

$$\omega = dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n$$

and $R = \frac{\partial}{\partial z}$, and the G action is just a linear action on the $2n$-dimensional vector space spanned by the first $2n$ coordinates.

4.2 Local Convexity

In this section, we prove a local convexity theorem for Hamiltonian torus actions on odd dimensional symplectic manifolds. We can think of this as a convexity theorem on the basic symplectic manifolds. As mentioned before, the Reeb flows can be very complicated, so the quotient basic symplectic manifolds can be non-Hausdorff. However, convexity properties still hold.

In the convexity theorem for ordinary symplectic manifolds, the group action is assumed to be effective. We will assume an extra tangency condition in the odd case: we require that for every G-orbit N and every $p \in N$, $R_p \notin T_pN$. The following remark explains how this can be viewed as effectiveness of the G action. Moreover, we will see an example in which the group action is effective but the convexity fails.

Remark 4.2.1. *The non-tangency condition means that the action is not along the direction of the Reeb flows on the basic symplectic manifold, being the quotient of M*

by this flow.

Even when G is a torus and the action is effective, if we do not require this non-tangency condition, the convexity property can fail. Here is an example.

Example 4.2.2 (parabola image). *Consider S^3 lying in the complex 2-plane $\mathbb{C}^2$ as $|z_1|^2 + |z_2|^2 = 1$. We use the polar coordinates $r_1, \theta_1, r_2, \theta_2$, where $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$, so that S^3 is defined by $r_1^2 + r_2^2 = 1$. Let $\Omega = d(r_1^2) \wedge d\theta_1 \wedge d\theta_2$ be the volume form and*

$$R = r_1^2 \frac{\partial}{\partial \theta_1} + r_2^2 \frac{\partial}{\partial \theta_2}$$

the Reeb vector field.

Note $\mathcal{L}_R \Omega = 0$, then

$$\omega = \iota(R)\Omega = -r_1^2 d(r_1^2) \wedge d\theta_2 + (1 - r_1^2) d(r_1^2) \wedge d\theta_1$$

is automatically a closed 2-form, and hence a symplectic 2-form. There is a T^2-action which rotates the angle coordinates θ_1 and θ_2. It is easy to check that this is a Hamiltonian action with a moment map

$$\Psi = \left(\frac{r_1^4}{2} - r_1^2, \frac{r_1^4}{2}\right),$$

where $0 \le r_1^2 \le 1$. The image of the moment map in the xy-plane is defined by the equation $(y - x)^2 = 2y$, $0 \le y \le \frac{1}{2}$, which is a parabola.

It is clear that the non-tangency condition fails, as expected.

Suppose M is a $(2n+1)$-dimensional symplectic manifold with a Hamiltonian G-action and $p \in M$. Let $\mathcal{O}_p$ be the G-orbit of p. By Theorem 4.1.4, there exists a G-invariant codimension one submanifold W containing $\mathcal{O}_p$ such that a neighborhood of $\mathcal{O}_p$ is equivariantly symplectomorphic to $W \times (-\varepsilon, \varepsilon)$ and the Reeb vector field is just $\frac{\partial}{\partial t}$. Since ω is a basic form, it is really a 2-form on W, and we use ω_W to denote. It is clear that ω_W is symplectic. Therefore, we can use the symplectic model developed in the even case to find the local structure of an odd dimensional symplectic manifold. Consequently, we obtain the local convexity for odd dimensional symplectic manifolds.

Suppose that G is a compact connected Lie group acting on a manifold M, and H is the stabilizer group at some point $p \in M$. Then we let M^H be the subspace of points fixed by H. Let M_H be the subspace of points with stabilizer group H, i.e.,

$$M^H = \{p \in M \mid h \cdot p = p, \text{ for all } h \in H\}$$

$$M_H = \{p \in M \mid G_p = H\}$$

We next state a standard result about the Lie group actions on manifolds without proof.

Proposition 4.2.3. *The connected components of M_H and M^H are smooth submanifolds of M. Moreover, M_H is an open subset of M^H.*

Now we come back to focus on Lie group actions on odd dimensional symplectic manifolds.

Proposition 4.2.4. *Suppose G is a compact connected Lie group and M is a Hamiltonian odd-dimensional symplectic G-manifold with a moment map Ψ. Then M^H itself is an odd dimensional symplectic submanifold of M and hence so is M_H.*

Proof. Since H preserves R, H has a symplectic representation on the symplectic vector space $(T_pM/R(p), \omega_p)$. Therefore, $(T_pM/R_p)^H$ is a symplectic vector subspace of $(T_pM/R(p), \omega_p)$. Note $R(p) \in (T_pM)^H = T_p(M^H)$, and thus $(T_pM/R_p)^H \cong (T_pM)^H/R_p = T_pM^H/R_p$ is a symplectic vector subspace. Therefore, M_H is an odd dimensional symplectic submanifold of M. $\square$

The following is the well-known local convexity theorem for Hamiltonian torus actions on ordinary symplectic manifolds.

Theorem 4.2.5 (Local Convexity Theorem for Hamiltonian Torus Actions on Symplectic Manifolds). *Let $\sigma : T \times M \to M$ be a Hamiltonian action of a torus T on a symplectic manifold M and $m_0 \in M$. Then there exists an arbitrarily small open* *neighborhood $\mathcal{U}$ of m_0 and a polyhedral cone $C_{m_0} \subset \mathfrak{t}^*$ with vertex $\Psi(m_0)$ such that the following is true:*

1. *$\Psi(\mathcal{U})$ is an open neighborhood of $\Psi(m_0)$ in C_{m_0};*

2. *$\Psi : \mathcal{U} \to C_{m_0}$ is an open map;*

3. *$\Psi^{-1}(\Psi(u)) \cap \mathcal{U}$ is connected for all $u \in \mathcal{U}$.*

We will next prove a local convexity theorem for Hamiltonian torus actions on odd dimensional symplectic manifolds.

Corollary 4.2.6 (Local Convexity Theorem for Hamiltonian Actions on Odd Dimensional Symplectic Manifolds). *Let $\sigma : T \times M \to M$ be a Hamiltonian action of a torus T on an odd dimensional symplectic manifold M and $m_0 \in M$. Suppose the action is nowhere tangent to the Reeb flows. Then there exists an arbitrarily small open neighborhood $\mathcal{U}$ of m_0 and a polyhedral cone $C_{m_0} \subset \mathfrak{t}^*$ with vertex $\Psi(m_0)$ such that the following is true:*

1. *$\Psi(\mathcal{U})$ is an open neighborhood of $\Psi(m_0)$ in C_{m_0};*

2. *$\Psi : \mathcal{U} \to C_{m_0}$ is an open map;*

3. *$\Psi^{-1}(\Psi(u)) \cap \mathcal{U}$ is connected for all $u \in \mathcal{U}$.*

Proof. For any $m_0 \in M$, by equivariant Darboux theorem 4.1.4 we have a codimension 1 open submanifold W and $w_0 \in W$ such that locally around $m_0 = (w_0, 0)$ the manifold is like $W \times (-\varepsilon, \varepsilon)$, and the torus T only acts on the W part. Moreover, there exists a symplectic 2-form ω_0 on the submanifold W such that $\pi^*\omega_0 = \omega$, where $\pi : W \times (-\varepsilon, \varepsilon) \to W$ is the natural projection.
Claim: $\mathcal{L}(R)\Psi = 0$, i.e., Ψ does not depend on the $(-\varepsilon, \varepsilon)$ part. Indeed, for any $X \in \mathfrak{t}$,

$$\langle \mathcal{L}(R)\Psi, X\rangle = \iota(R)d\langle \Psi, X\rangle = \iota(R)\iota(X_M)\omega = 0$$

Therefore, Ψ induces a map $\Phi : W \to \mathfrak{t}^*$ such that for any $w \in W$, $z \in (-\varepsilon, \varepsilon)$

$$\Psi(w, z) = \Phi(w)$$

or equivalently, $\Psi = \pi^*\Phi$.

Note that for any $X \in \mathfrak{t}$,

$$\pi^*(\iota(\pi_* X_M)\omega_0) = \iota(X_M)(\omega) = d\langle \Psi, X\rangle = \pi^*(d\langle \Phi, X\rangle)$$

Therefore, $\iota(\pi_* X_M)\omega_0 = d\langle \Phi, X\rangle$, i.e., T acts on W in a Hamiltonian fashion and Φ is an associated moment map.

By theorem 4.2.5, there exists an arbitrarily small open neighborhood $\mathcal{W}$ of w_0 in W and a polyhedral cone $C_{w_0} \subset \mathfrak{t}^*$ with vertex $\Phi(w_0)$ such that the following is true:

1. $\Phi(\mathcal{W})$ is an open neighborhood of $\Phi(w_0)$ in C_{w_0};

2. $\Phi : W \to C_{w_0}$ is an open map;

3. $\Phi^{-1}(\Phi(w)) \cap \mathcal{W}$ is connected for all $w \in \mathcal{W}$.

For $m_0 = (w_0, 0)$, we let the arbitrarily small open neighborhood be $\mathcal{U} := \mathcal{W} \times (-\epsilon, \epsilon)$ where $\epsilon < \varepsilon$, and let the polyhedral cone be $C_{m_0} := C_{w_0}$ with vertex $\Phi(w_0) = \Psi(m_0)$. Now let's prove the three statements.

1. $\Psi(\mathcal{U}) = \Phi(\mathcal{W})$ is an open neighborhood of $\Psi(m_0)(= \Phi(w_0))$ in C_{m_0}.

2. Note that $\pi : \mathcal{U} = \mathcal{W} \times (-\epsilon, \epsilon) \to \mathcal{W}$ is an open map and we know that Φ is an open map, so the composition map $\Psi = \Phi \circ \pi$ is open as well.

3. For any $u = (w, z) \in \mathcal{U}$

$$\begin{aligned} \Psi^{-1}(\Psi(u)) \cap \mathcal{U} &= \Psi^{-1}(\Phi(w)) \cap \mathcal{U} \\ &= (\Phi^{-1}(\Phi(w)) \times (-\epsilon, \epsilon)) \cap (\mathcal{W} \times (-\varepsilon, \varepsilon)) \\ &= (\Phi^{-1}(\Phi(w)) \cap \mathcal{W}) \times (-\epsilon, \epsilon) \end{aligned}$$

Therefore, $\Psi^{-1}(\Psi(u)) \cap \mathcal{U}$ is connected since $\Phi^{-1}(\Phi(w)) \cap \mathcal{W}$ is connected.

$\square$

4.3 Local-Global-Principle

In this section, we use the Local-Global-Principle to prove a convexity theorem for Hamiltonian torus actions on odd dimensional symplectic manifold. This Local-Global-Principle was initiated by Condevaux-Dazord-Molino in [CDM] and developed in [HNP1], [BOR1] and [BK] by Hilgert-Neeb-Plank, Birtea-Ortega-Ratiu, and Bjorndahl-Karshon. It can be viewed as a generalization of Tietze-Nakajima Theorem, see [T] and [N] .

In [L1] Lerman used this technique to prove a convexity theorem for torus actions on contact manifolds, in which he assumed that the contact distribution is transverse to torus orbits. In [CK] Chiang-Karshon studied convexity package for moment maps and generalized Lerman's convexity theorem. In the last section, we get the local convexity on odd dimensional symplectic manifolds under the non-tangency assumption, which is somewhat the opposite to Lerman's transversality condition. It will be interesting to see whether there is a relation between these two convexity theorems.

We next introduce some notions which will be used to describe the Local-Global-Principle. The notions we use here follow Birtea-Ortega-Ratiu's paper. First we need the following concepts.

Definition 4.3.1. *Let X and Y be two topological spaces and $f : X \to Y$ a continuous map. The subset $A \subset X$ satisfies the* **locally fiber connected condition (LFC)** *if A does not intersect two different connected components of the fiber $f^{-1}f(x)$, for any $x \in A$.*

Let X be a connected, locally connected, Hausdorff topological space and V a locally convex topological vector space. The continuous map $f : X \to V$ is said to be **locally fiber connected** *if for each U_x satisfies* **(LFC)** *condition.*

Definition 4.3.2. *A map $\Psi : X \to V$ is said to have* **local convexity data** *if for each $x \in X$ there exists an arbitrarily small open neighborhood U_x of x and a convex cone C_x with vertex $\Psi(x)$ in V such that $\Psi(U_x)$ is a neighborhood of the vertex $\Psi(x)$ in C_x and such that $\Psi|_{U_x} : U_x \to C_x$ is an open map, where C_x is endowed with the subspace topology from V.*

Now we are ready to state the Local-Global-Principle. For the proof of the principle, we refer the reader to [BOR1], in which there is a very detailed description.

Theorem 4.3.3 (Local-Global-Principle). *Let $\Psi : X \to V$ be a locally fiber connected map from a connected locally connected Hausdorff topological space to a finite dimensional vector space, with local convexity data $(C_x)_{x\in M}$. Suppose Ψ is proper. Then $\Psi(M)$ is a closed locally polyhedral convex subset of V, the fiber $\Psi^{-1}(v)$ is connected for each $v \in V$, $\Psi : X \to \Psi(X)$ is an open mapping, and*

$$C_x = \{\Psi(x) + \lambda(\Psi(y) - \Psi(x)) |\ y \in X, \lambda \geq 0\}$$

holds for all $x \in X$.

Theorem 4.3.4 (Global Convexity). *Let $\sigma : T \times M \to M$ be a Hamiltonian action of a torus T on a compact odd-dimensional symplectic manifold M which is transverse to the Reeb flows and $m_0 \in M$. Then the image of the moment map is a rational simple convex polytope Δ. Moreover, all the fibers $\Psi^{-1}(v)$ are connected $v \in \mathfrak{t}^*$.*

Proof. This is obvious by the local convexity and the Local-Global-Principle. □

We conclude this chapter by further studying the structure of moment polytope Δ. Let $H \subset T$ be the isotropy group of some point $p \in M$, and M_H be the space of points with isotropy group H.

Recall that in the Proposition 4.2.4 we prove that M_H is an open subset of the symplectic submanifold M^H. Each connected component of M^H is a Hamiltonian odd-dimensional symplectic T-manifold in its own right, with H acting trivially. Thus its moment map image is a convex polytope of dimension $\dim(T/H)$ inside an affine subspace $\mu+\mathrm{ann}(\mathfrak{h})$, with the corresponding component of M^H mapping to its interior. That is, the open faces of Δ correspond to orbit type strata, and in particular the vertices of Δ correspond to fixed points M^T. Note however that some of the polytopes $\Psi(M^H)$ get mapped to the interior of Δ. Thus Δ gets subdivided into polyhedral subregions, consisting of regular values of Ψ.

Theorem 4.3.5. *Suppose (M, ω) is a Hamiltonian T-manifold with a moment map Ψ. Suppose T acts effectively on M. Let $\Delta \subset \mathfrak{t}^*$ be the rational convex moment polytope. Then the closed faces Σ of Δ of codimension d correspond to a closed Lie subgroup H of dimension d such that $\Psi^{-1}(\Sigma)$ is an odd-dimensional symplectic submanifold of M, and is a connected component of M^H, and Σ lies in the affine subspace ann$(\mathfrak{h})$. In particular, the preimage of the vertices of Δ lies in M^T, that is*

$$\Delta = Hull(\Psi(M^T))$$

Proof. We only need to prove $\Psi^{-1}(\Sigma)$ is connected, which is obvious since all the fibers of the moment map Ψ is connected by Theorem 4.3.4. $\square$

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